EE 508
Lecture 11

The Approximation Problem

Classical Approximating Functions

- Elliptic Approximations
- Thompson and Bessel Approximations

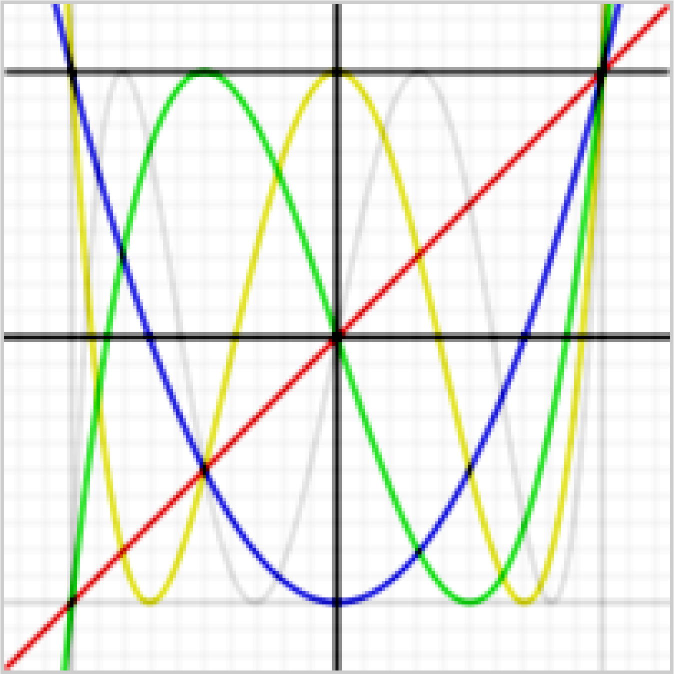
Review from Last Time

Chebyshev Approximations

Chebyshev Polynomials

The first 9 CC polynomials:

$$C_0(x) = 1$$
$$C_1(x) = x$$
$$C_2(x) = 2x^2 - 1$$
$$C_3(x) = 4x^3 - 3x$$
$$C_4(x) = 8x^4 - 8x^2 + 1$$
$$C_5(x) = 16x^5 - 20x^3 + 5x$$
$$C_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$C_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$
$$C_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$



This image shows the first few Chebyshev polynomials of the first kind in the domain $-1\frac{1}{4} < x < 1\frac{1}{4}$, $-1\frac{1}{4} < y < 1\frac{1}{4}$; the flat T_0, and T_1, T_2, T_3, T_4 and T_5. Figure from Wikipedia

- Even-indexed polynomials are functions of x^2
- Odd-indexed polynomials are product of x and function of x^2
- Square of all polynomials are function of x^2 (i.e. an even function of x)

Review from Last Time

Chebyshev Approximations

Type 1

$$H_{BW}(\omega)=\frac{1}{1+\varepsilon^2\omega^{2n}}$$

Butterworth

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_n\left(\omega^2\right)}$$

A General Form

Observation:

$F_n(\omega^2)$ close to 1 in the pass band and gets very large in stop-band

The square of the Chebyshev polynomials have this property

$$H_{CC}(\omega)=\frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

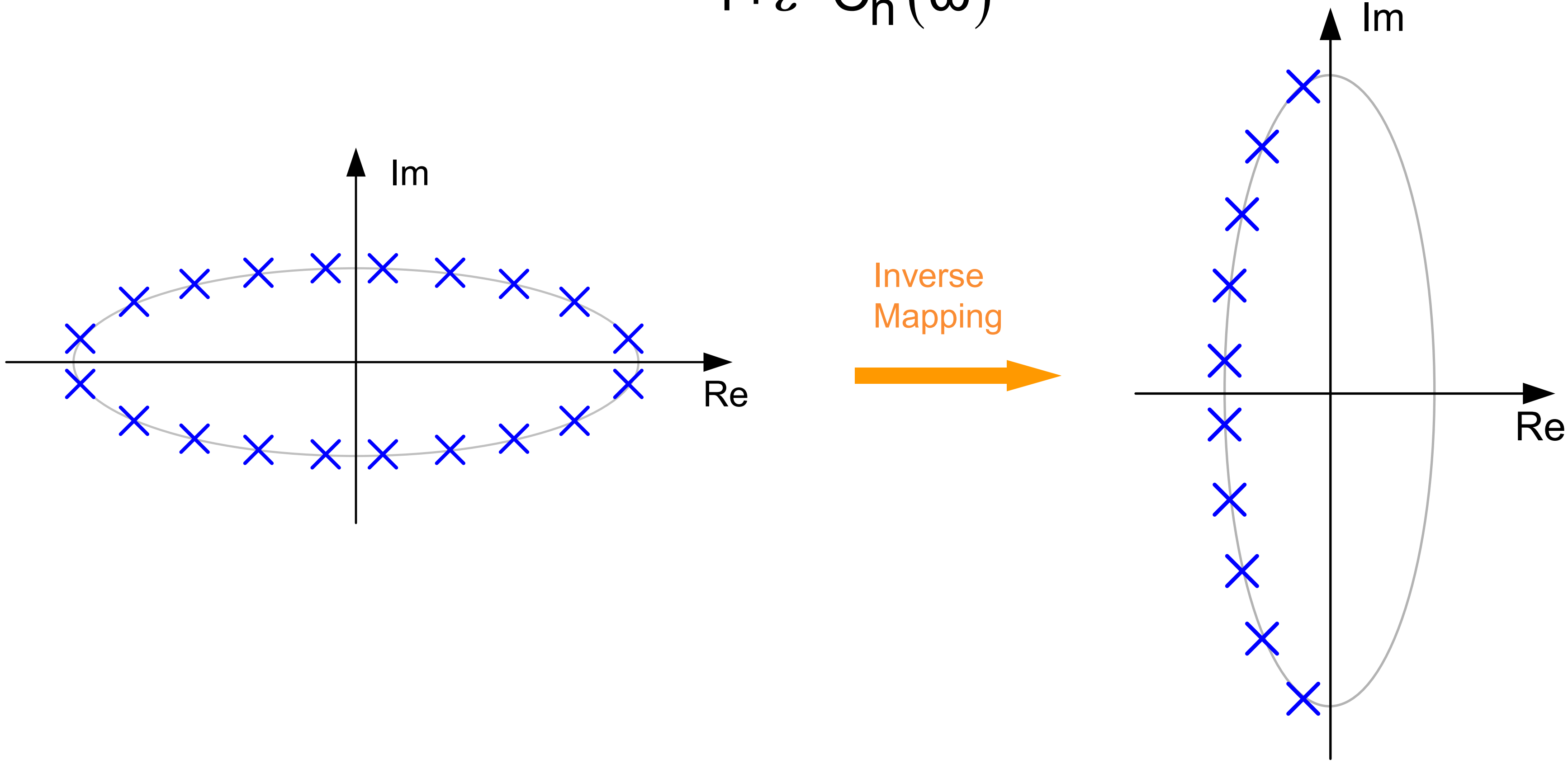
This is the magnitude squared approximating function of the Type 1 CC approximation

Review from Last Time

Chebyshev Approximations

Type 1

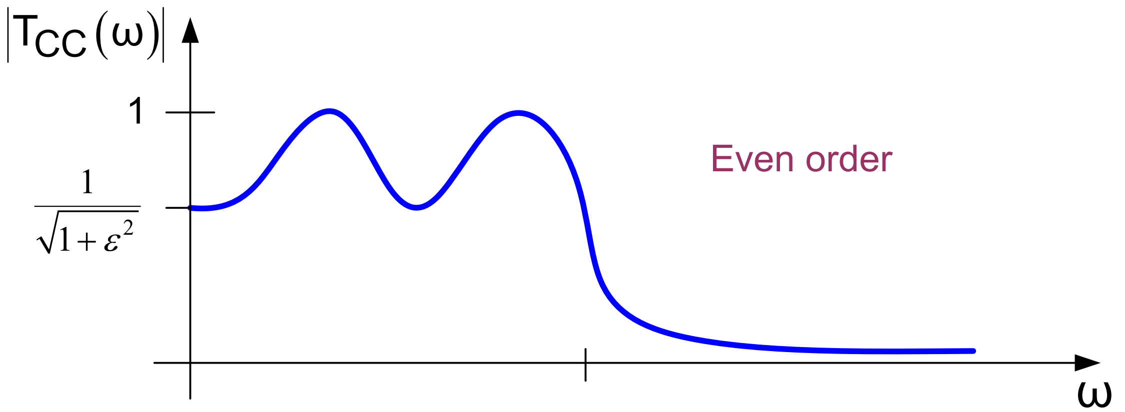
$$H_{CC}(\omega) = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

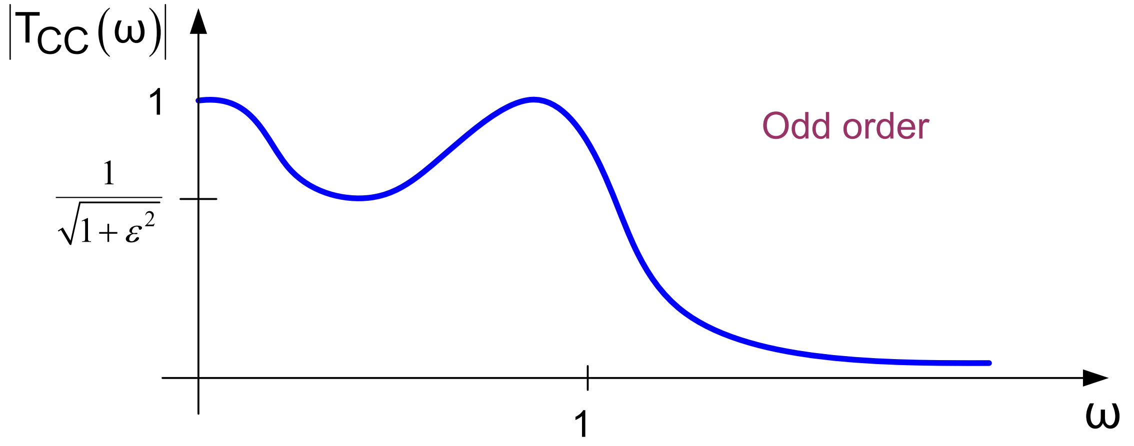


Poles of $T_{CC}(s)$

Chebyshev Approximations

Type 1

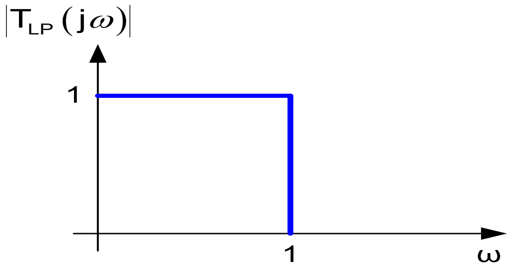




- $|T_{CC}(0)|$ alternates between 1 and $\sqrt{\frac{1}{1+\varepsilon^2}}$ with index number
- Substantial pass band ripple
- Sharp transitions from pass band to stop band

Review from Last Time

Chebyshev Approximations



Type II Chebyshev Approximations (not so common)

- Analytical formulation:
  - Magnitude response bounded between 0 and $\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ in the stop band
  - Assumes the value of $\sqrt{\frac{1}{1+\varepsilon^2}}$ at ω=1
  - Value of 1 at ω=0
  - Assumes extreme values maximum times in [1 ∞]
  - Characterized by {n,ε}
- Based upon Chebyshev Polynomials

Review from Last Time

Comparison of BW and Type 1 CC Responses

- CC slope at band edge much steeper than that of BW

$$Slope_{CC}(\omega=1)=\left(\frac{-n}{2\sqrt{2}}\right)n=n\bullet[Slope_{BW}(\omega=1)]$$

- Corresponding pole Q of CC much higher than that of BW
- Lower-order CC filter can often meet same band-edge transition as a given BW filter
- Both are widely used
- Cost of implementation of BW and CC for same order is about the same

Transitional BW-Chebyshev Approximations

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_n\left(\omega^2\right)}$$

General Form

Define $F_{BWk}=\omega^{2k}$ $F_{CCk}=C_n^2(\omega)$

Consider:

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_{BWk}F_{CC(n\text{-}k)}} \qquad 0 \le k \le n$$

$$H(\omega)=\frac{1}{1+\varepsilon^2\left[(\theta)F_{BWk}+(1-\theta)F_{CC(n\text{-}k)}\right]} \qquad 0 \le \theta \le 1$$

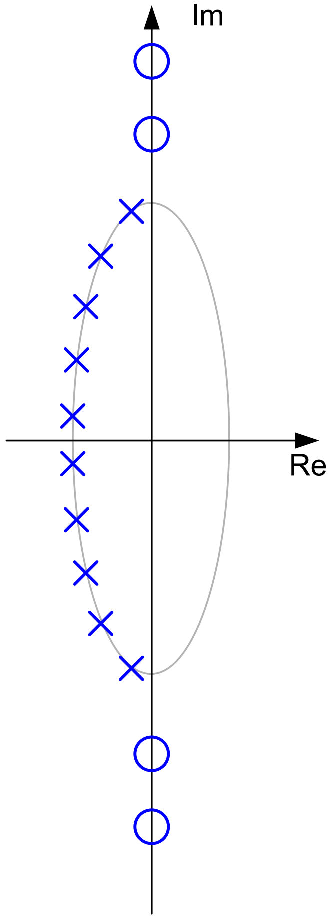
- Other transitional approximations are possible
- Transitional approximations have some of the properties of both “parents”

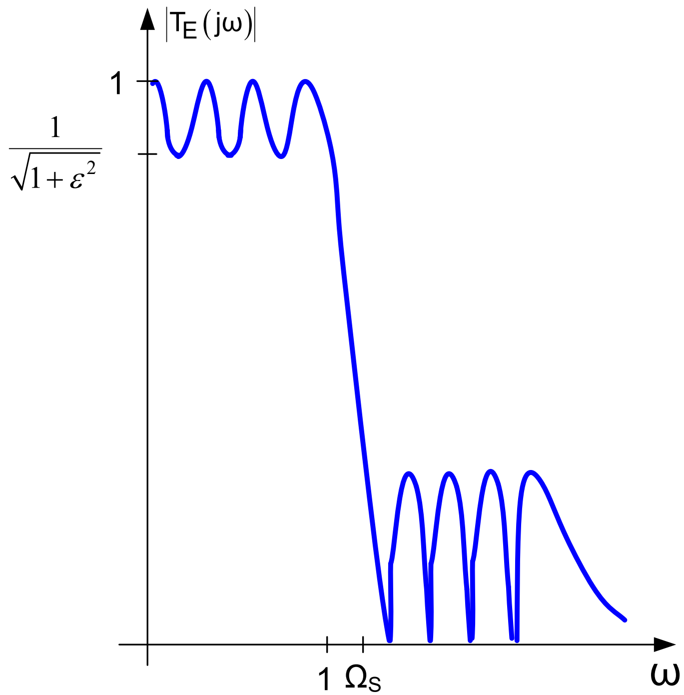
Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyshev
  - Elliptic
  - Bessel
  - Thompson

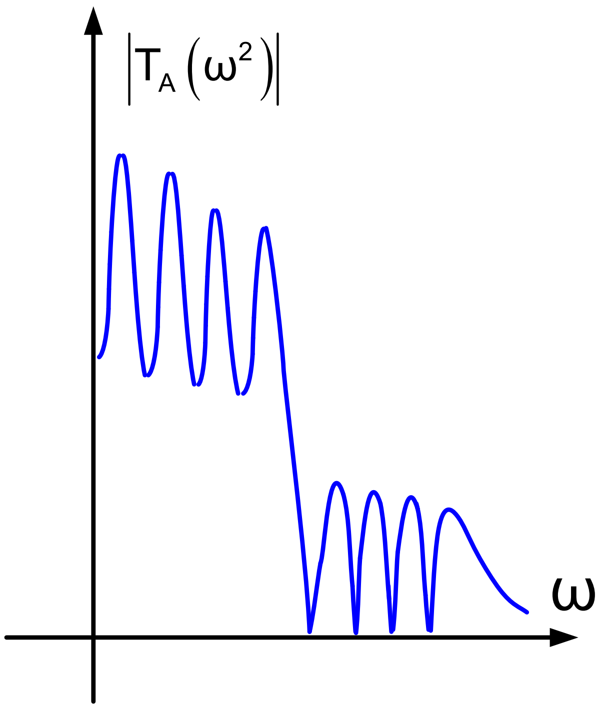
Elliptic Filters

Can be thought of as an extension of the CC approach by adding complex-conjugate zeros on the imaginary axis to increase the sharpness of the slope at the band edge





Concept



Actual effect of adding zeros

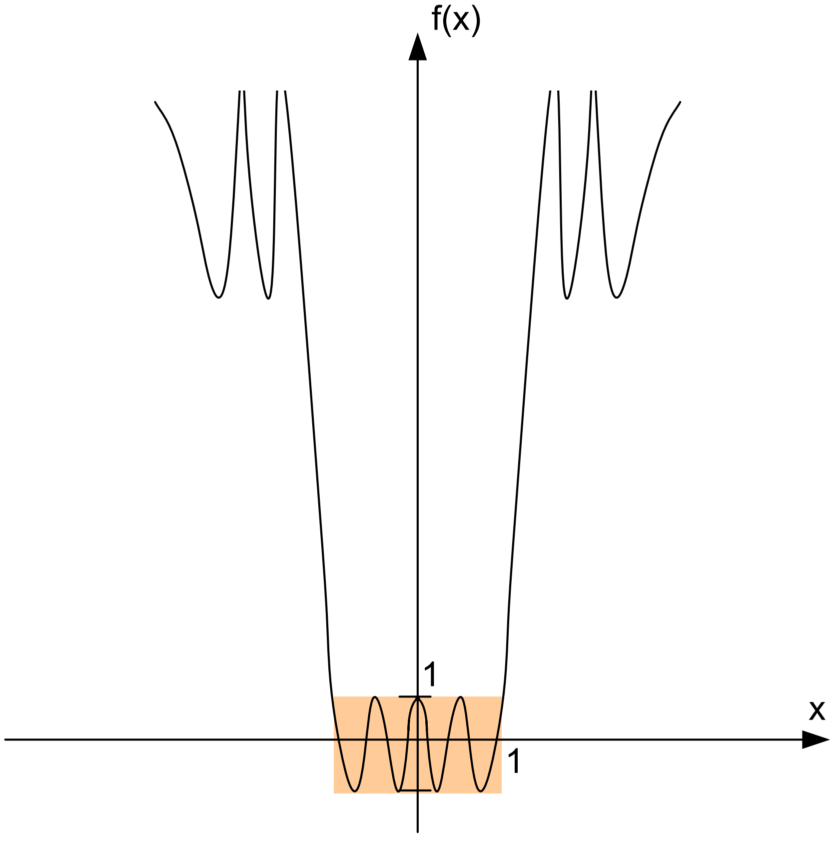
Elliptic Filters

- Basic idea comes from the concept of a Chebyschev Rational Fraction
- Sometimes termed Cauer filters

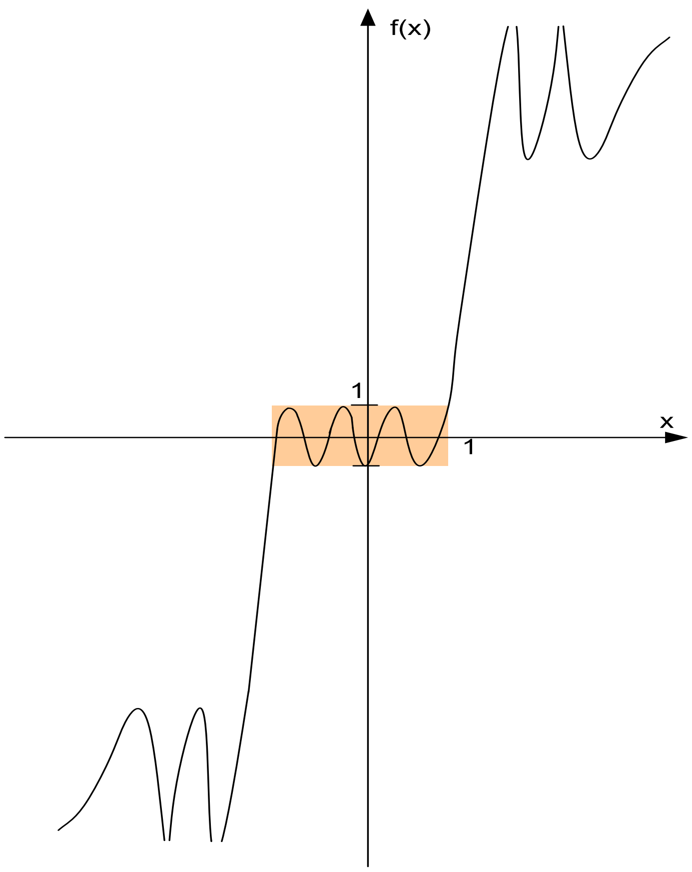
Chebyshev Rational Fraction

A Chebyshev Rational Fraction is a rational fraction that is equal ripple in [-1,1] and equal ripple in [-∞,-1] and [1,∞]

Chebyshev Rational Fractions



Even-order CC rational fraction



Odd-order CC rational fraction

Chebyshev Rational Fractions

Even-order CC rational fraction

$$C_{Rn}(x) = H \frac{\prod_{k=1}^{n/2}\left(x^2 - a_k\right)}{\prod_{k=1}^{n/2}\left(x^2 - b_k\right)}$$

Odd-order CC rational fraction

$$C_{Rn}(x) = H \frac{x\prod_{k=1}^{\frac{n-1}{2}}\left(x^2 - a_k\right)}{\prod_{k=1}^{\frac{n-1}{2}}\left(x^2 - b_k\right)}$$

Elliptic Filters

Magnitude-Squared Elliptic Approximating Function

$$H_E(\omega) = \frac{1}{1+\varepsilon^2 C_{Rn}^2(\omega)}$$

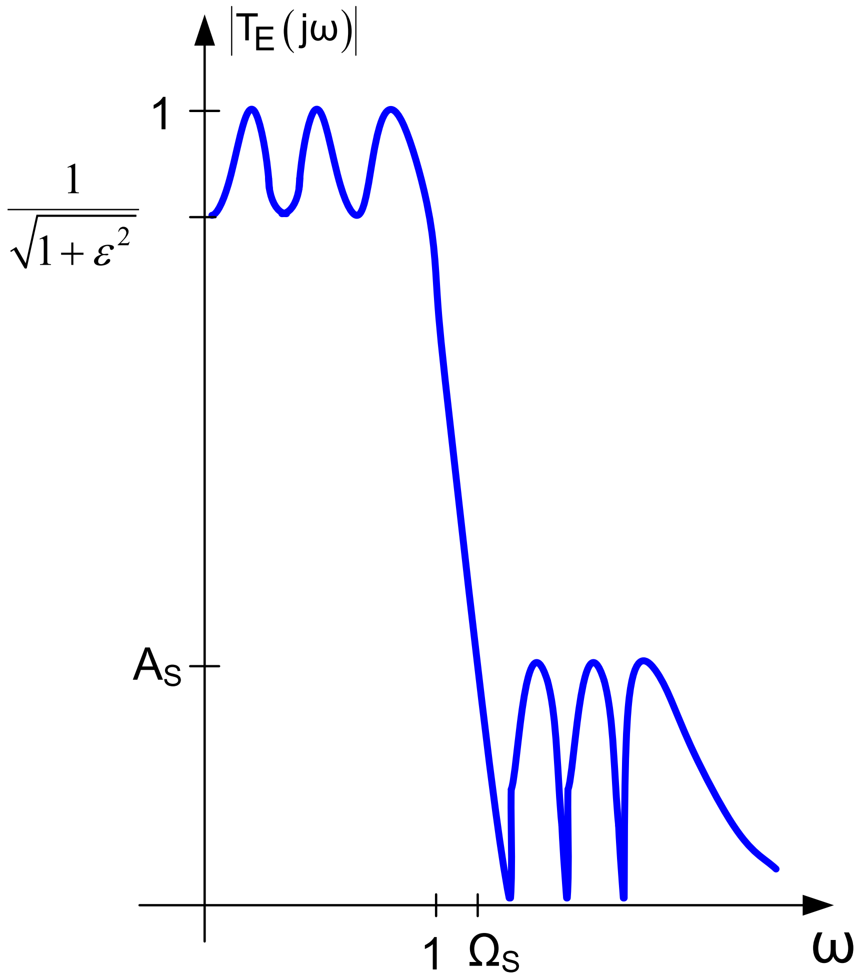
Inverse mapping to $T_E(s)$ exists

- For n even, n zeros on imaginary axis
- For n odd, n-1 zeros on imaginary axis
- Equal ripple in both pass band and stop band
- Analytical expression for poles and zeros not available
- Often choose to have less than n or n-1 zeros on imaginary axis

Termed here "full order" (refers to the first two items)

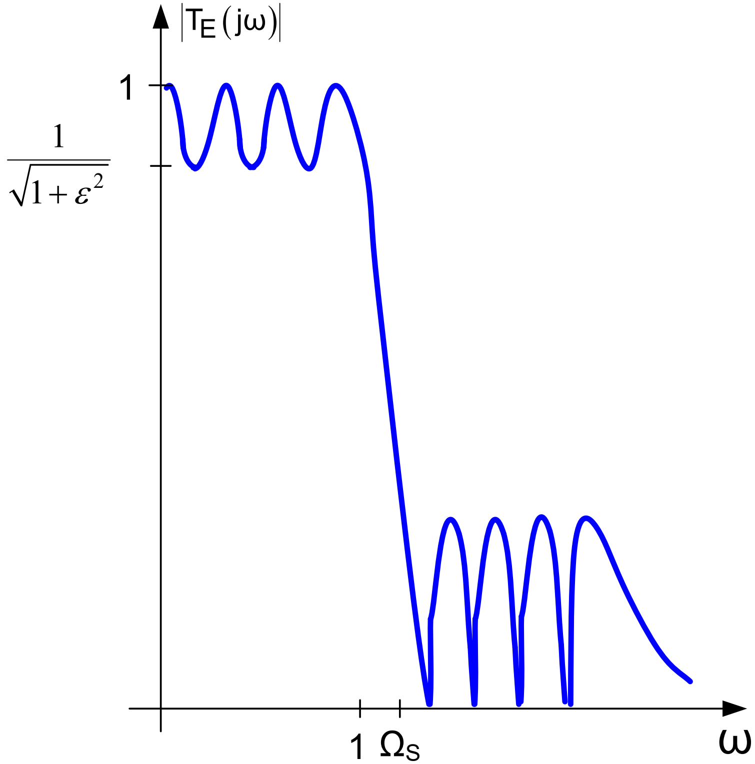
(No longer based upon CC rational fractions)

Elliptic Filters

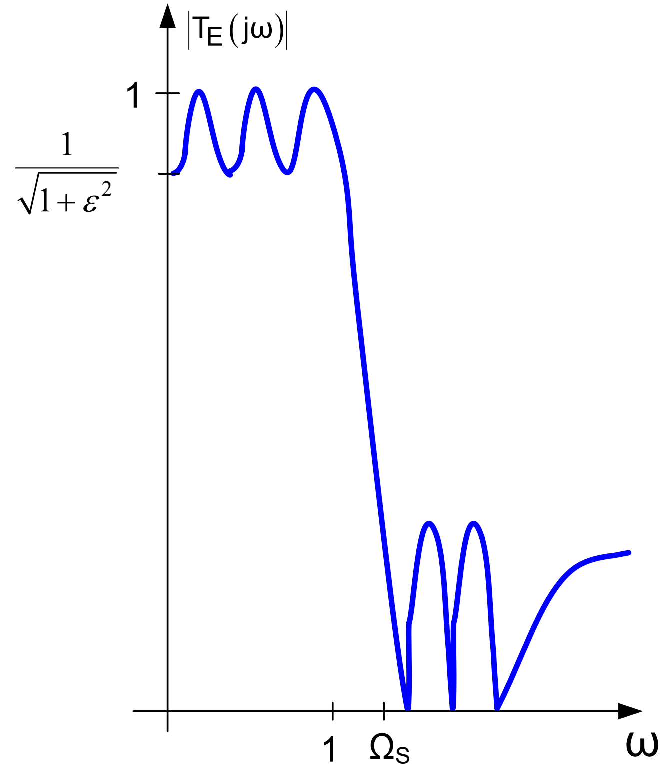


- If of full-order, response completely characterized by {n,ε,A_S.Ω_S}
- Any 3 of these paramaters are independent
- Typically ε,Ω_S, and A_S are fixed by specifications (i.e. must determine n)

Elliptic Filters



n odd



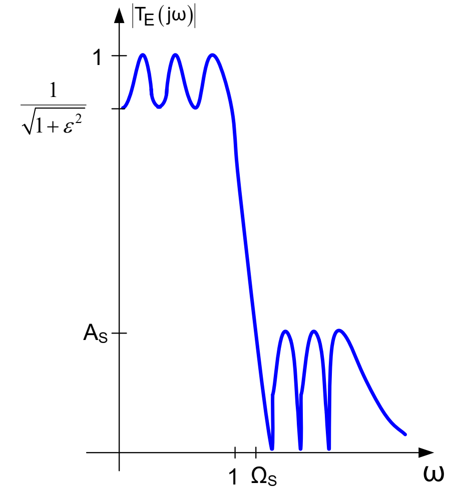
n even

For full-order elliptic approximations

- (n-1)/2 peaks in pass band
- (n-1)/2 peaks in stop band
- Maximum occurs at ω=0
- $|T(j\infty)|=0$

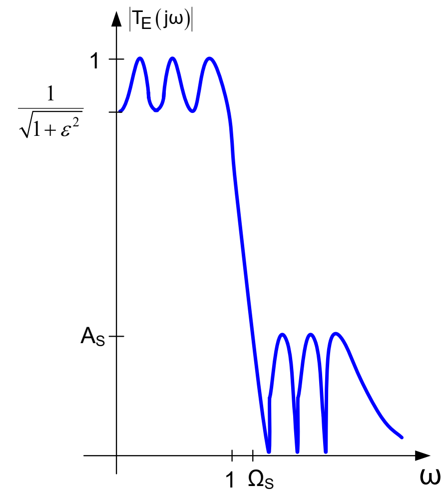
- n/2 peaks in pass band
- n/2 peaks in stop band
- $|T(j0)|=1/\text{sq}(1+\varepsilon^2)$
- $|T(j\infty)|=A_S$

Elliptic Filters



- Simple closed-form expressions for poles, zeros, and $|T_E(j\omega)|$ do not exist
- Simple closed form expressions for relationship between {n,ε,AS, and Ω_S} do not exist
- Simple expressions for max pole Q and slope at band edge do not exist
- Reduced-order elliptic approximations could be viewed as CC filters with zeros added to stop band
- General design tables not available though limited tables for specific characterization parameters do exist

Elliptic Filters



Observations about Elliptic Filters

- Elliptic filters have steeper transitions than CC1 filters
- Elliptic filters do not roll off as quickly in stop band as CC1 or even BW
- Highest Pole-Q of elliptic filters is larger than that of CC filters
- For a given transition requirement, order of elliptic filter typically less than that of CC filter
- Cost of implementing elliptic filter is comparable to that of CC filter if orders are the same
- Cost of implementing a given filter requirement is often less with the elliptic filters
- Often need computer to obtain elliptic approximating functions though limited tables are available
- Some authors refer to elliptic filters as Cauer filters

Canonical Approximating Functions

- Butterworth
- Chebyschev
- Transitional BW-CC
- Elliptic
- → Thomson
- Bessel

Thompson and Bessel Approximating Functions are Two Different Names for the Same Approximation

Background in Bessel/Thomson filters

The filter's name is a reference to German mathematician Friedrich Bessel (1784–1846), who developed the mathematical theory on which the filter is based. The filters are also called **Bessel–Thomson filters** in recognition of W. E. Thomson, who worked out how to apply Bessel functions to filter design in 1949.[2]

Thomson, W. E. (November 1949). "Delay networks having maximally flat frequency characteristics" (PDF). *Proceedings of the IEE - Part III: Radio and Communication Engineering*. **96** (44): 487–490.

Thomson and Bessel Approximations

- All-pole filters
- Maximally linear phase at $\omega=0$

Thompson and Bessel Approximations

Consider T(jω)

$$T(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_R(j\omega)+jN_{IM}(j\omega)}{D_R(j\omega)+jD_{IM}(j\omega)}$$

$$\text{phase} = \angle\big(T(j\omega)\big) = \tan^{-1}\left(\frac{N_I(j\omega)}{N_R(j\omega)}\right) - \tan^{-1}\left(\frac{D_I(j\omega)}{D_R(j\omega)}\right)$$

- Phase expressions are very difficult to work with !!
- Will first consider group delay and frequency distortion

Linear Phase

Consider T(jω)

$$T(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_R(j\omega)+jN_{IM}(j\omega)}{D_R(j\omega)+jD_{IM}(j\omega)}$$

$$\angle\left(T\left(j\omega\right)\right) = \tan^{-1}\left(\frac{N_I\left(j\omega\right)}{N_R\left(j\omega\right)}\right) - \tan^{-1}\left(\frac{D_I\left(j\omega\right)}{D_R\left(j\omega\right)}\right)$$

Defn: A filter is said to have linear phase if the phase is given by the expression

$\angle\left(T\left(j\omega\right)\right) = \theta\omega$ where θ is a constant that is independent of ω

Note: Linear phase definition requires not only linear relationship but must pass through the origin

Distortion in Filters

Types of Distortion

Frequency Distortion

- Amplitude Distortion
- Phase Distortion

Nonlinear Distortion

Although the term "distortion" is used for these two basic classes, there is little in common between these two classes

Distortion in Filters

Frequency Distortion

- Amplitude Distortion

  A filter is said to have frequency (magnitude) distortion if the magnitude of the transfer function changes with frequency

- Phase Distortion

  A filter is said to have phase distortion if the phase of the transfer function is not equal to a constant times ω

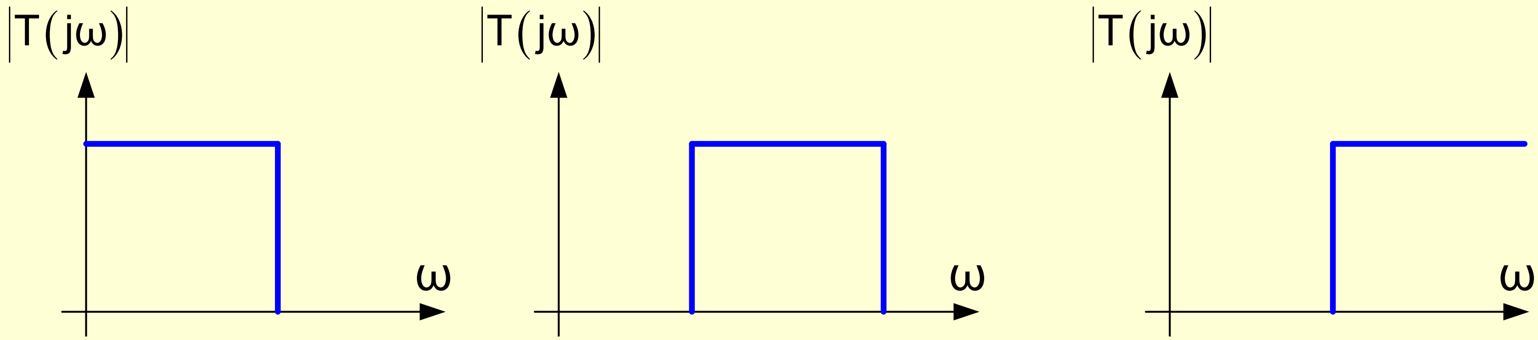
Nonlinear Distortion

A filter is said to have nonlinear distortion if there is one or more spectral components in the output that are not present in the input

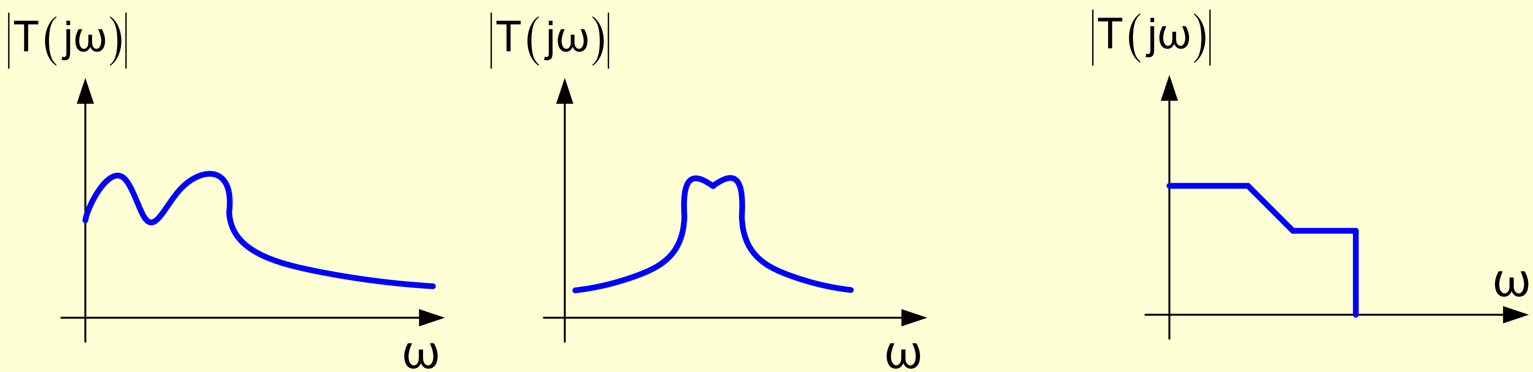
Distortion in Filters

- Phase and frequency distortion are concepts that apply to linear circuits
- If frequency distortion is present, the relative magnitude of the spectral components that are present in the output will be different than the spectral components in the input
- If phase distortion is present, at least for some inputs, waveshape will not be preserved
- Nonlinear distortion does not exist in linear networks and is often used as a measure of the linearity of a filter.
- No magnitude distortion will be present in a specific output of a filter if all spectral components that are present in the input are in a flat passband
- No phase distortion will be present in a specific output of a filter if all spectral components that are present in the input are in a linear phase passband
- Linear phase can occur even when the magnitude in the passband is not flat
- Linear phase will still occur if the phase becomes nonlinear in the stopband

Filter Passband and Stopband



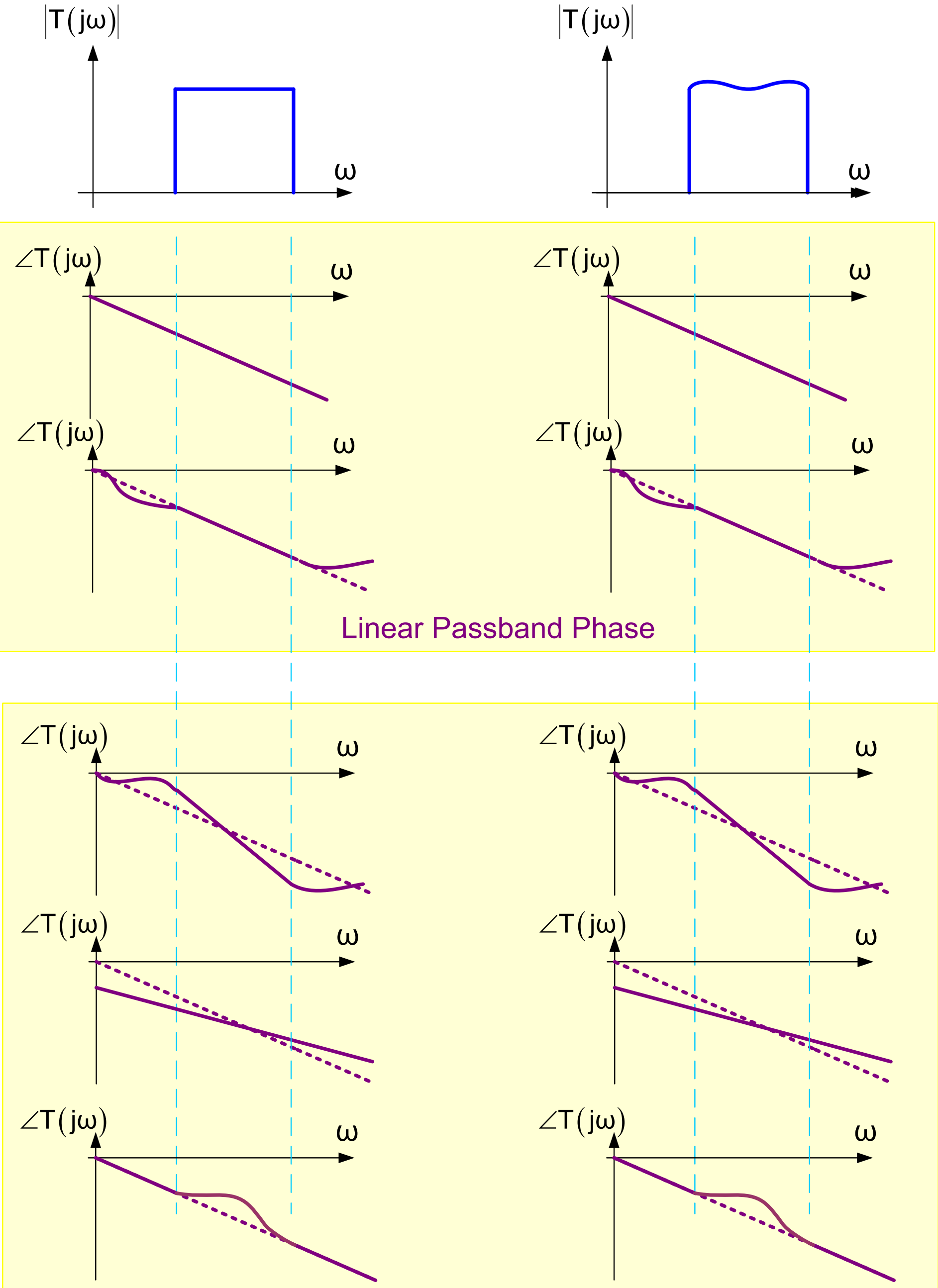
Flat Passband



Non-Flat Passband

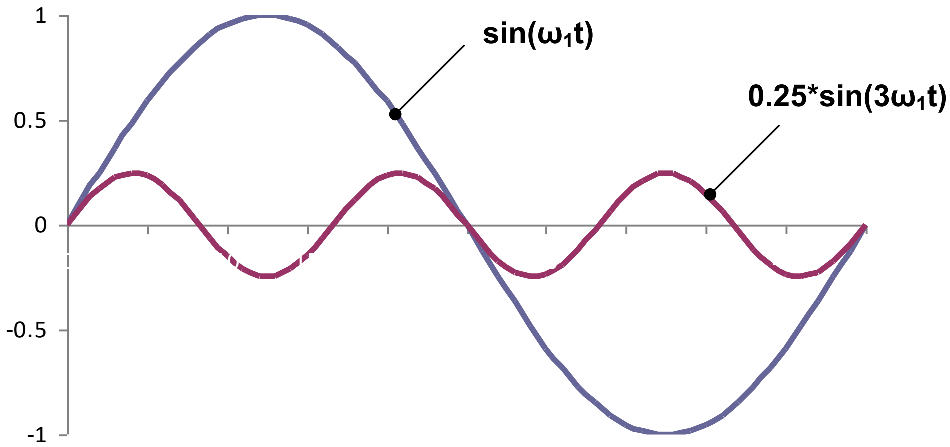
- Frequencies where gain is ideally 0 or very small is termed the stopband
- Frequencies where the gain is ideally not small is termed the passband
- Passband is often a continuous region in ω though could be split

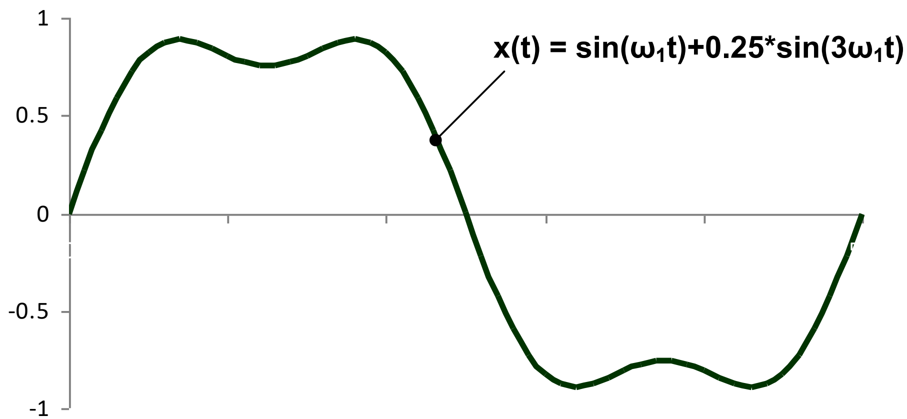
Linear and Nonlinear Phase

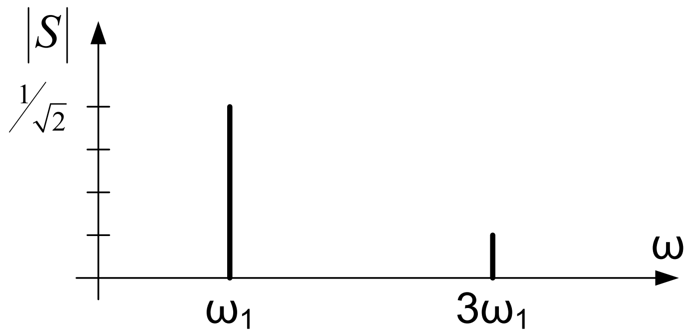


Preserving the Waveshape:

Example: Consider a signal $x(t)=\sin(\omega_1 t) + 0.25\sin(3\omega_1 t)$







Note the wave shape and spectral magnitude of x(t)

Preserving the waveshape

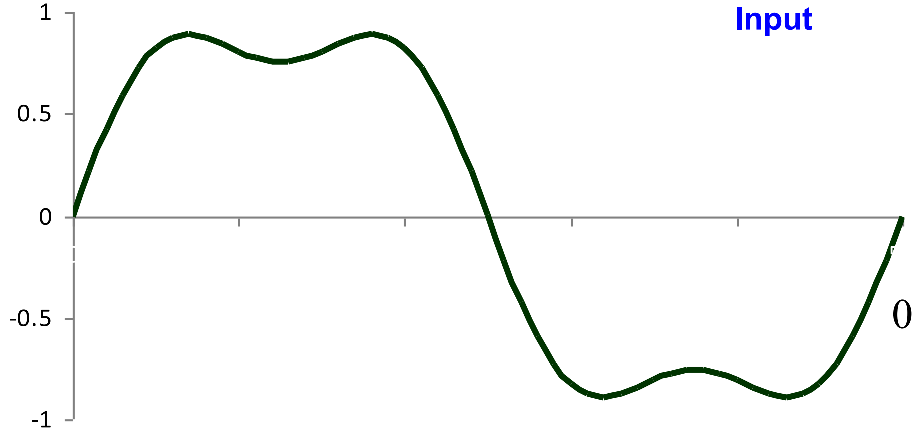
A filter has no frequency distortion for a given input if the output wave shape is preserved (i.e. the output wave shape is a magnitude scaled and possibly time-shifted version of the input)

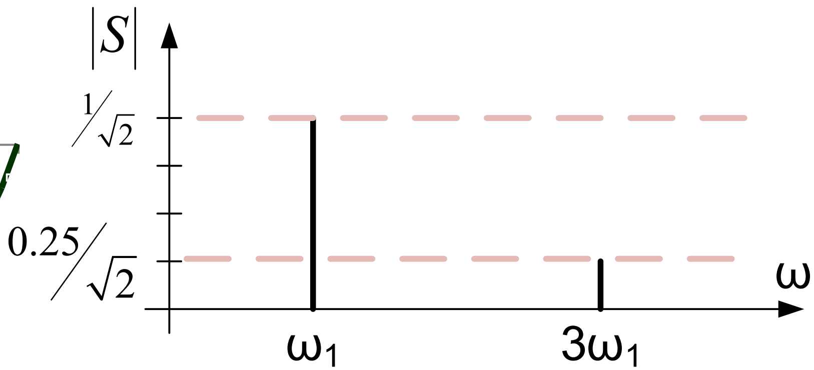
Mathematically, no frequency distortion for $V_{IN}(t)$ if

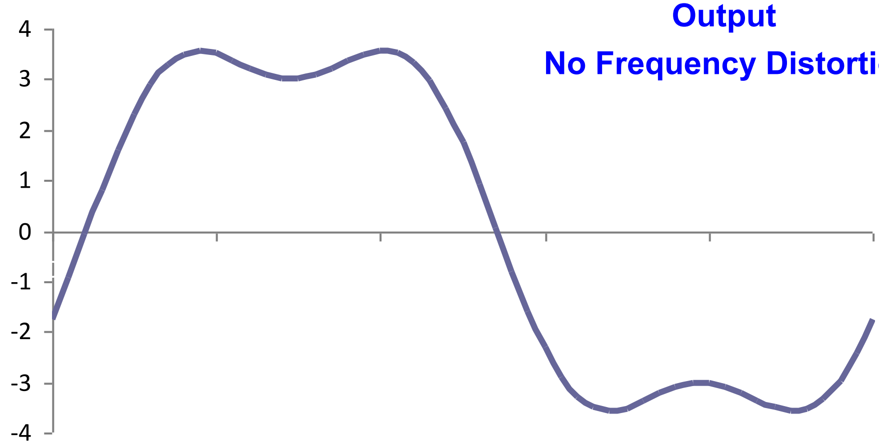
$$V_{OUT}(t)=KV_{IN}(t-t_{shift})$$

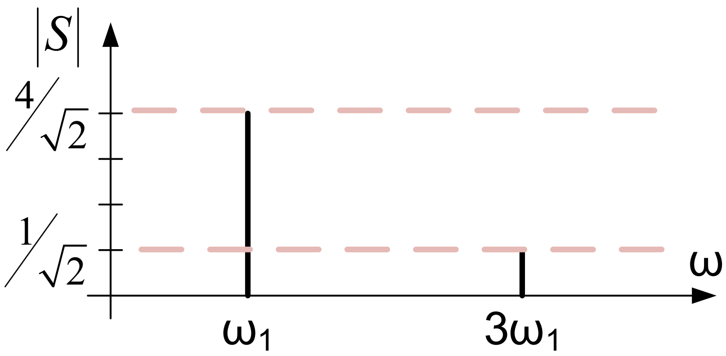
Could have frequency distortion for other inputs

Example of No Frequency Distortion



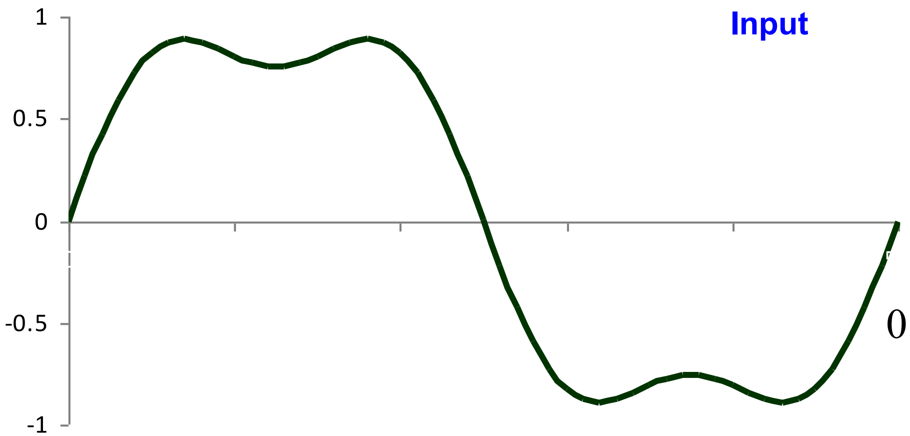


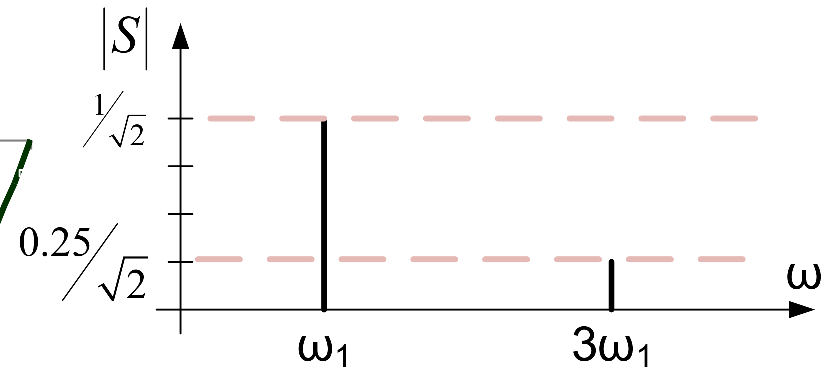


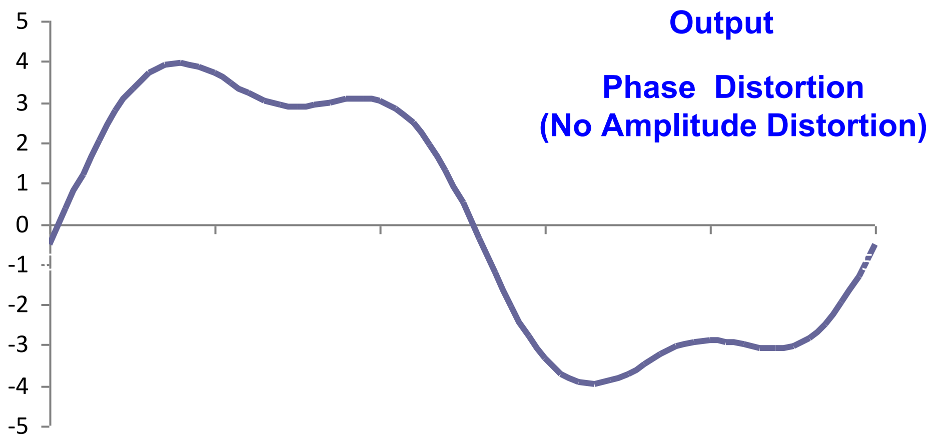


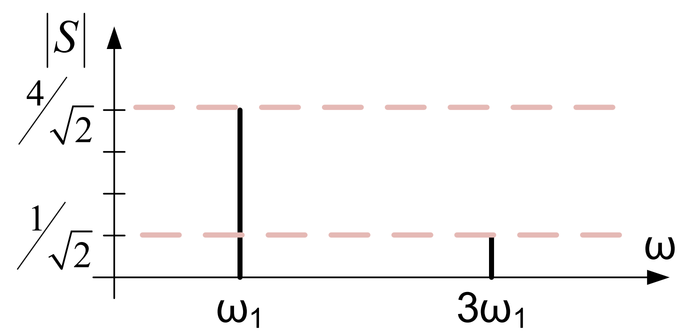
Waveshape Preserved

Example with Frequency Distortion



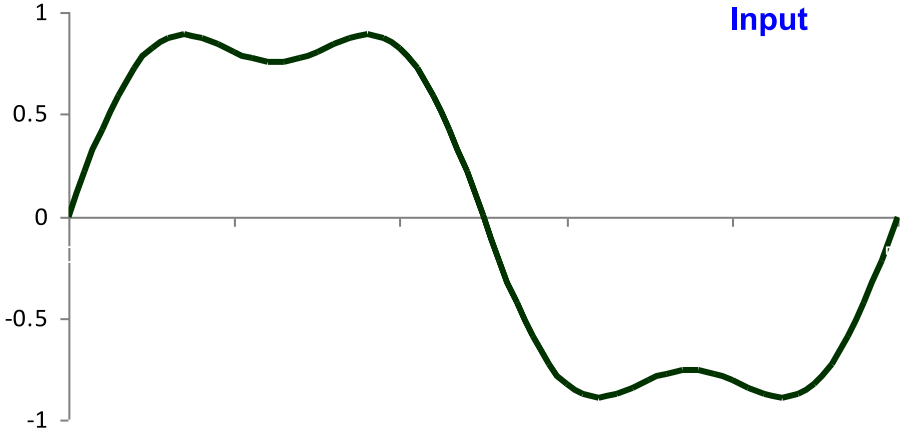


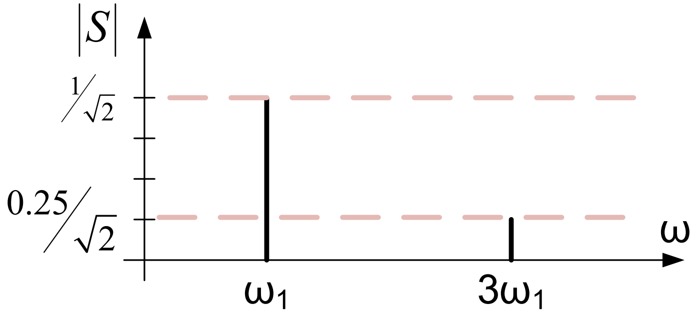


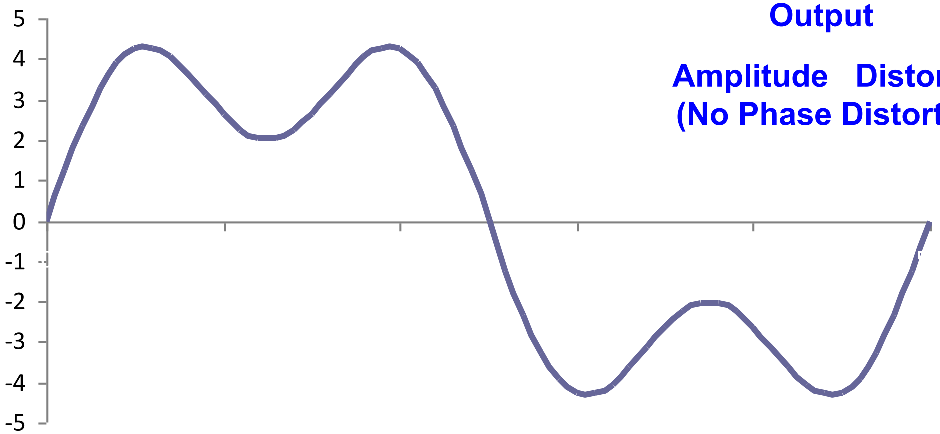


Waveshape Not Preserved

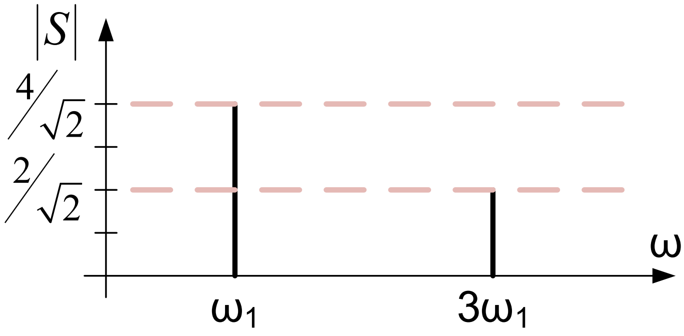
Example with Frequency Distortion





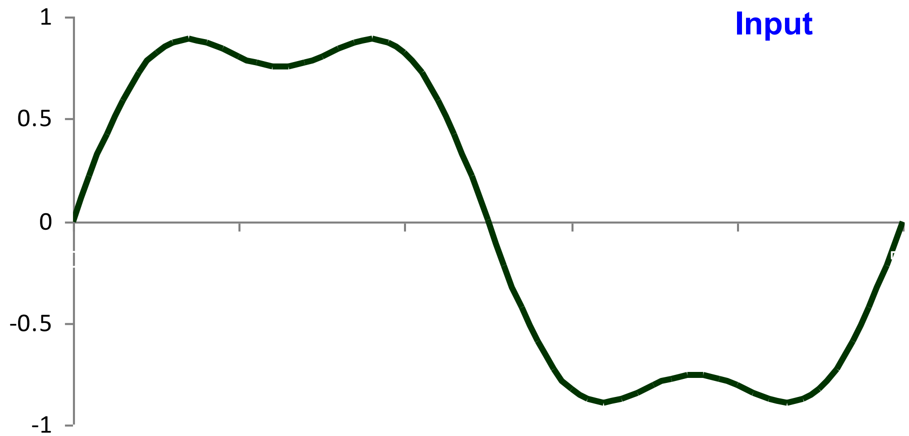


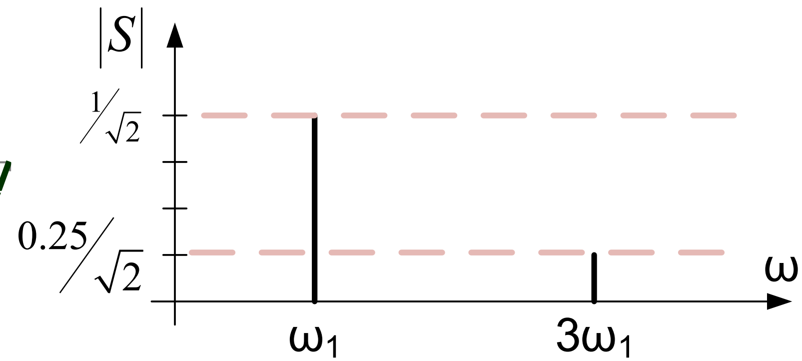
Amplitude Distortion
(No Phase Distortion)

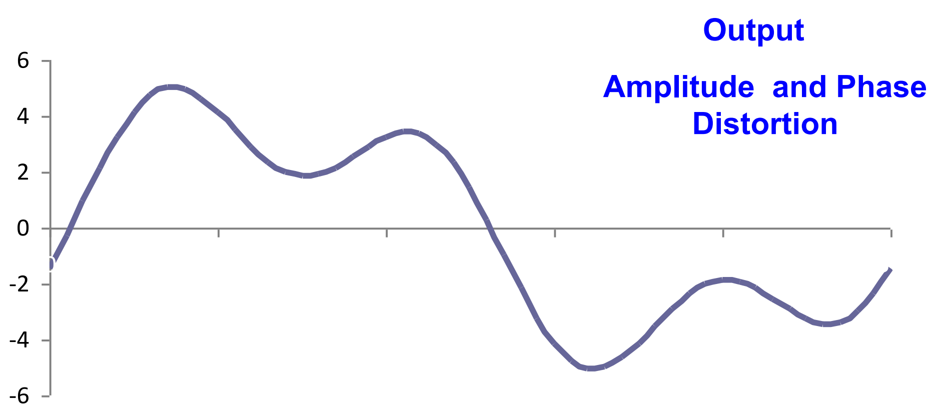


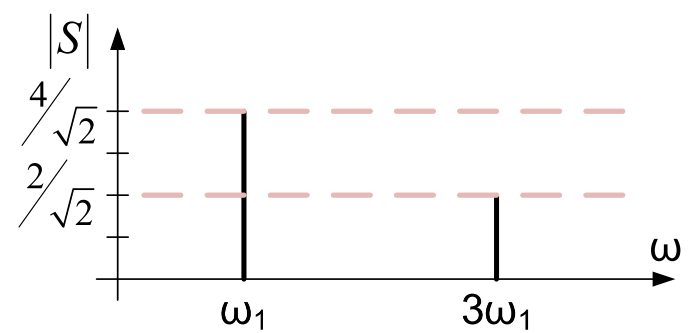
Waveshape Not Preserved

Example with Frequency Distortion



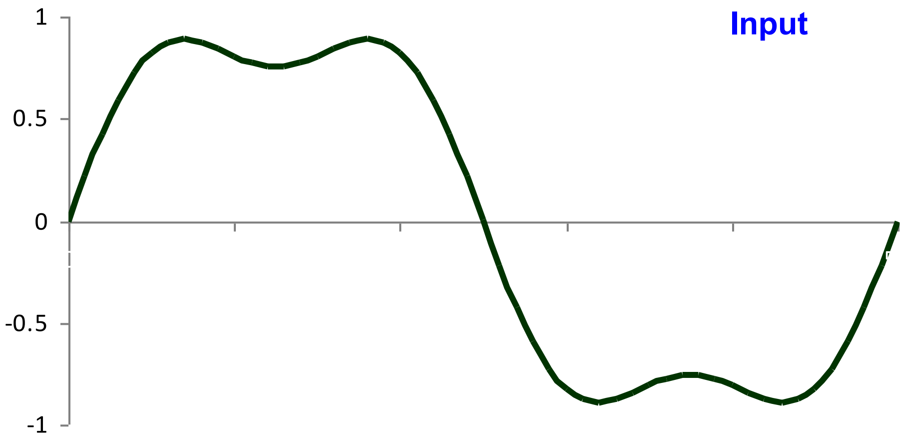


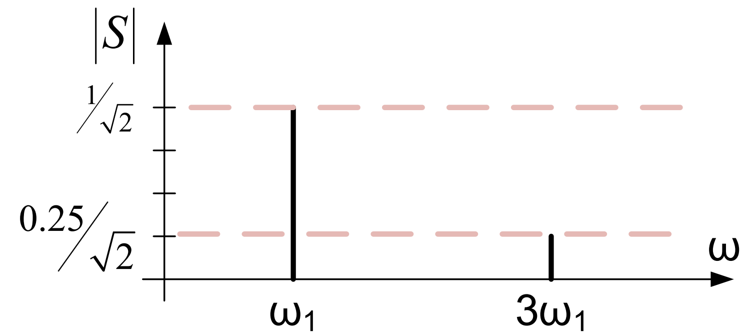


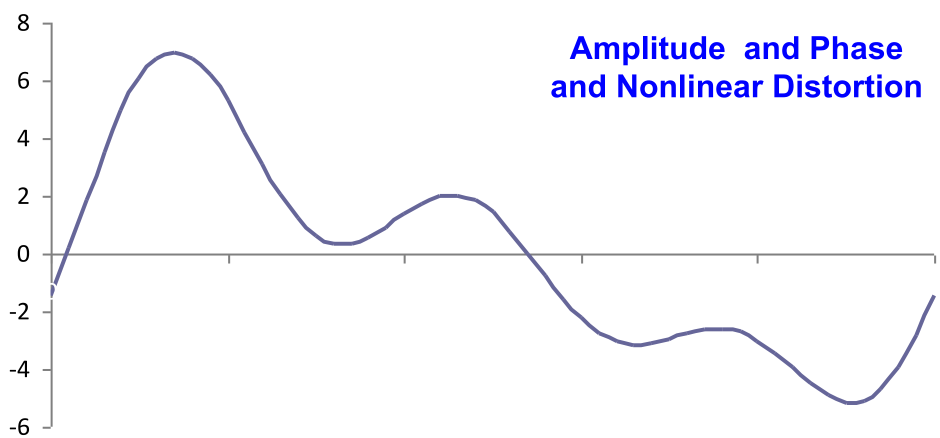


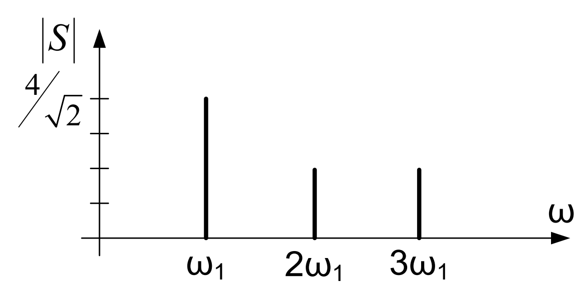
Waveshape Not Preserved

Example with Nonlinear Distortion and Frequency Distortion









Waveshape Not Preserved

Nonlinear distortion evidenced by presence of spectral components in output that are not in the input

Frequency Distortion

In most audio applications (and many other signal processing applications) there is little concern about phase distortion but some applications do require low phase distortion

In audio applications, any substantive magnitude distortion in the pass band is usually not acceptable

Any substantive nonlinear distortion in the pass band is unacceptable in most audio applications

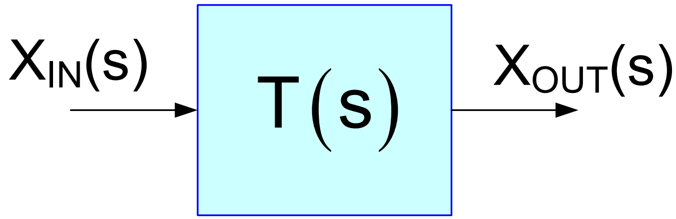
Preserving wave-shape in pass band

A filter is said to have linear passband phase if the phase in the passband of the filter is given by the expression $\angle\left(T\left(j\omega\right)\right)=\theta\omega$ where θ is a constant that is independent of ω

If a filter has linear passband phase in a flat passband, then the waveshape is preserved provided all spectral components of the input are in the passband and the output can be expressed as an amplitude scaled and time shifted version of the input by the expression

$$V_{OUT}(t)=KV_{IN}(t-t_{shift})$$

Preserving wave-shape in pass band



Example:

Consider a linear network with transfer function T(s)

Assume $$X_{in}(t) = A_1\sin(\omega_1 t+\theta_1) + A_2\sin(\omega_2 t+\theta_2)$$

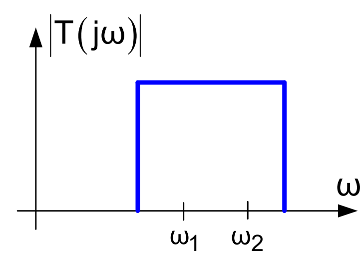
In the steady state

$$X_{OUT}(t) = A_1\left|T(j\omega_1)\right|\sin\left(\omega_1 t+\theta_1+\angle T(j\omega_1)\right) + A_2\left|T(j\omega_2)\right|\sin\left(\omega_2 t+\theta_2+\angle T(j\omega_2)\right)$$

Rewrite as:

$$X_{OUT}(t) = A_1\left|T(j\omega_1)\right|\sin\left(\omega_1\left[t+\frac{\angle T(j\omega_1)}{\omega_1}\right]+\theta_1\right) + A_2\left|T(j\omega_2)\right|\sin\left(\omega_2\left[t+\frac{\angle T(j\omega_2)}{\omega_2}\right]+\theta_2\right)$$

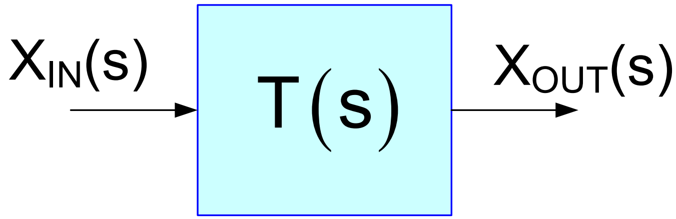
If ω_1 and ω_2 are in a flat passband, $\left|T(j\omega_1)\right| = \left|T(j\omega_2)\right|$



Can express as:

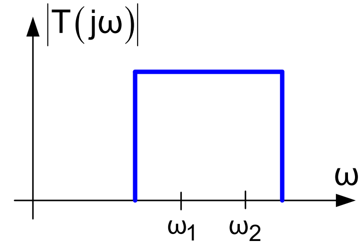
$$X_{OUT}(t) = \left|T(j\omega_1)\right|\left\{A_1\sin\left(\omega_1\left[t+\frac{\angle T(j\omega_1)}{\omega_1}\right]+\theta_1\right) + A_2\sin\left(\omega_2\left[t+\frac{\angle T(j\omega_2)}{\omega_2}\right]+\theta_2\right)\right\}$$

Preserving wave-shape in pass band



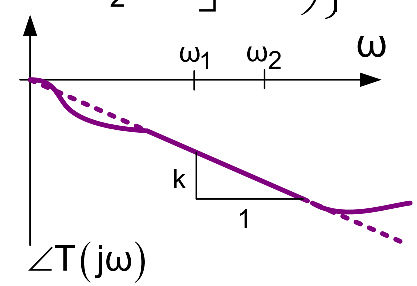
Example:

$$X_{in}(t) = A_1\sin(\omega_1 t+\theta_1) + A_2\sin(\omega_2 t+\theta_2)$$



If ω_1 and ω_2 are in a flat passband, $|T(j\omega_1)| = |T(j\omega_2)|$

$$X_{OUT}(t) = |T(j\omega_1)|\left\{A_1\sin\left(\omega_1\left[t+\frac{\angle T(j\omega_1)}{\omega_1}\right]+\theta_1\right) + A_2\sin\left(\omega_2\left[t+\frac{\angle T(j\omega_2)}{\omega_2}\right]+\theta_2\right)\right\}$$



If ω_1 and ω_2 are in a linear phase passband, $\angle T(j\omega) = k\omega$

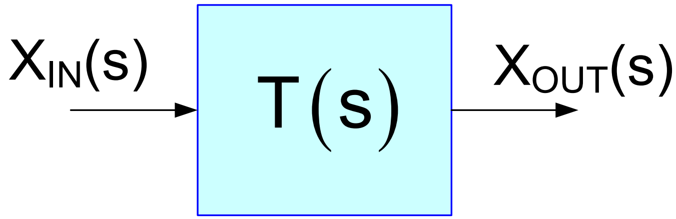
$$\angle T(j\omega_1) = k\omega_1 \quad \text{and} \quad \angle T(j\omega_2) = k\omega_2$$

$$X_{OUT}(t) = |T(j\omega_1)|\left\{A_1\sin\left(\omega_1\left[t+\frac{k\omega_1}{\omega_1}\right]+\theta_1\right) + A_2\sin\left(\omega_2\left[t+\frac{k\omega_2}{\omega_2}\right]+\theta_2\right)\right\}$$

$$X_{OUT}(t) = |T(j\omega_1)|\left\{A_1\sin(\omega_1[t+k]+\theta_1) + A_2\sin(\omega_2[t+k]+\theta_2)\right\}$$

$$X_{OUT}(t) = |T(j\omega_1)|\,x_{in}(t+k)$$

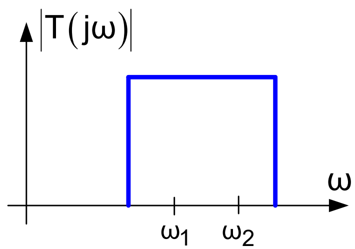
Preserving wave-shape in pass band



Example:

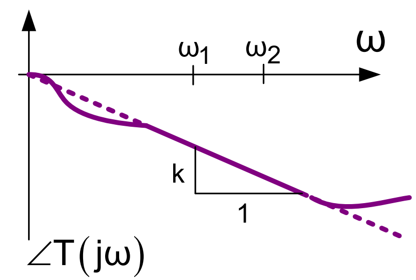
$$X_{in}(t) = A_1\sin(\omega_1 t+\theta_1) + A_2\sin(\omega_2 t+\theta_2)$$

$$|T(j\omega_1)| = |T(j\omega_2)|$$



$$\angle T(j\omega_1) = k\omega_1$$

$$\angle T(j\omega_2) = k\omega_2$$



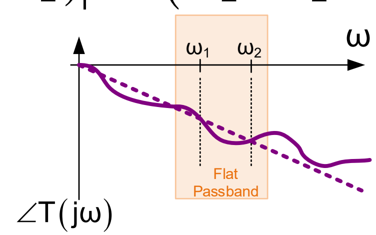
$$X_{OUT}(t) = |T(j\omega_1)|x_{in}(t+k)$$

This is a magnitude scaled and time shifted version of the input so waveshape is preserved

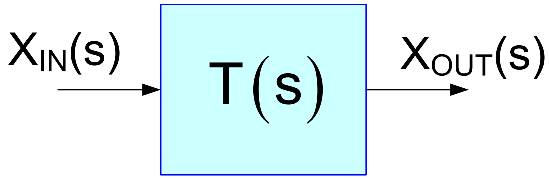
A weaker condition on the phase relationship will also preserve waveshape with two specific spectral components present

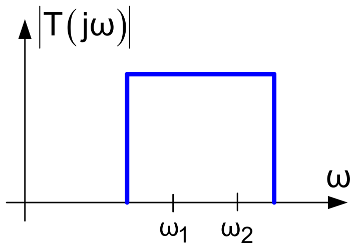
$$X_{OUT}(t) = A_1|T(j\omega_1)|\sin(\omega_1 t+\theta_1 + \angle T(j\omega_1)) + A_2|T(j\omega_2)|\sin(\omega_2 t+\theta_2 + \angle T(j\omega_2))$$

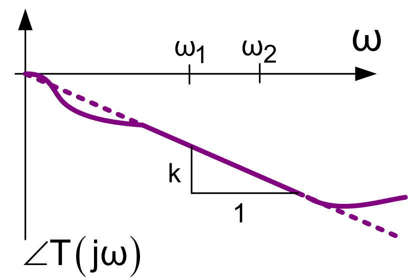
$$\frac{\angle T(j\omega_1)}{\angle T(j\omega_2)} = \frac{\omega_1}{\omega_2}$$



Amplitude (Magnitude) Distortion, Phase Distortion and Preserving wave-shape in pass band





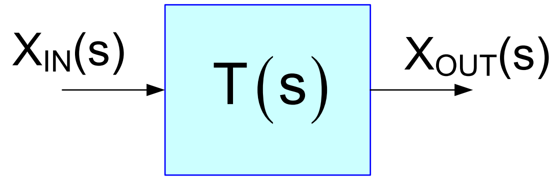


If ω_1 and ω_2 are any two spectral components of an input signal in which $\left|T(j\omega_1)\right| \neq \left|T(j\omega_2)\right|$ then the filter exhibits amplitude distortion for this input.

If ω_1 and ω_2 are any two spectral components of an input signal in which $\dfrac{\angle T(j\omega_1)}{\angle T(j\omega_2)} \neq \dfrac{\omega_1}{\omega_2}$ then the filter exhibits phase distortion for this input.

If ω_1 and ω_2 are any two spectral components of an input signal that exhibits either amplitude or phase distortion for these inputs, then the waveshape will not be preserved $X_{OUT}(t) \neq H \bullet x_{in}(t+k)$

Amplitude (Magnitude) Distortion, Phase Distortion and Preserving wave-shape in pass band



Amplitude and phase distortion are often of concern in filter applications requiring a flat passband and a flat zero-magnitude stop band

Amplitude distortion is usually of little concern in the stopband of a filter

Phase distortion is usually of little concern in the stopband of a filter

A filter with no amplitude distortion or phase distortion in the passband and a zero-magnitude stop band will exhibit waveform distortion for any input that has a frequency component in the passband and another frequency component in the stopband

It can be shown that the only way to avoid magnitude and phase distortion respectively for signals that have energy components in the interval $\omega_1 < \omega < \omega_2$ is to have constants k_1 and k_2 such that

$$\left.\begin{aligned} \left|T(j\omega)\right| &= k_1 \\ \angle T(j\omega) &= k_2\omega \end{aligned}\right\} \quad \text{for } \omega_1 < \omega < \omega_2$$

Group Delay

Defn: Group Delay is the negative of the phase derivative with respect to ω

$$\tau_G = -\frac{d\angle T(j\omega)}{d\omega}$$

Recall, by definition, the phase is linear iff $\angle T(j\omega) = k\omega$

If the phase is linear, $\tau_G = -\frac{d\angle T(j\omega)}{d\omega} = -\frac{d(k\omega)}{d\omega} = -k$

Thus for $\angle T(j0) = 0$, the phase is linear iff the group delay is constant

The group delay and the phase of a transfer function carry the same information

But, of what use is the group delay?

Group Delay

Example: Consider what is one of the simplest transfer functions

$$T(s)=\frac{1}{s+1}$$

$$T(j\omega)=\frac{1}{j\omega+1} \qquad \angle T(j\omega)= -\tan^{-1}\left(\frac{\omega}{1}\right) \qquad \tau_G=-\frac{d\angle T(j\omega)}{d\omega}$$

The phase of T(s) is analytically very complicated

$$\tau_G=-\frac{d\angle T(j\omega)}{d\omega}=-\frac{d\left(-\tan^{-1}\omega\right)}{d\omega}$$

Recall the identity

$$\frac{d\left(\tan^{-1}u\right)}{dx}=\left(\frac{1}{1+u^2}\right)\frac{du}{dx}$$

$$\tau_G=-\frac{d\left(-\tan^{-1}\omega\right)}{d\omega}=--\frac{1}{1+\omega^2}$$

Thus

$$\tau_G=\frac{1}{1+\omega^2}$$

Note that the group delay is a rational fraction in ω^2 instead of an arctan function

Group Delay

But, of what use is the group delay?

The phase of almost all useful transfer functions are complicated functions involving sums of arctan functions and these are difficult to work with analytically

Theorem: The group delay of any transfer function is a rational fraction in ω^2

But, of what use is the group delay?

Qualitatively:

The following two criteria are equivalent:

- Maximally linear phase at $\omega=0$
- Maximally constant group delay at $\omega=0$

Analytically working with the group delay (rational fraction in ω^2) rather than the phase (difference between 2 arctan functions) is much more mathematically tractable

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem (only shown here for case of all-pole transfer function):

$$T(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

$$T(j\omega) = \frac{1}{\left(1 - a_2\omega^2 + a_4\omega^4 + \ldots\right) + j\omega\left(a_1 - a_3\omega^2 + a_5\omega^4 + \ldots\right)}$$

$$T(j\omega) = \frac{1}{F_1\left(\omega^2\right) + j\omega F_2\left(\omega^2\right)}$$

where F_1 and F_2 are even polynomials in ω

$$\angle T(j\omega) = -\tan^{-1}\left(\frac{\omega F_2\left(\omega^2\right)}{F_1\left(\omega^2\right)}\right)$$

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem: $$\angle T(j\omega) = -\tan^{-1}\left(\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right)$$

but from identity $$\frac{d(\tan^{-1}u)}{dx} = \left(\frac{1}{1+u^2}\right)\frac{du}{dx}$$

$$\tau_G = -\frac{d\angle T(j\omega)}{d\omega} = -\frac{1}{1+\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]^2} \bullet \frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega}$$

Now consider the right-most term in the product

$$\frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega} = \frac{F_1(\omega^2)\left[\frac{d\left(\omega F_2(\omega^2)\right)}{d\omega}\right] - \left(\omega F_2(\omega^2)\right)\frac{d\left(F_1(\omega^2)\right)}{d\omega}}{\left[F_1(\omega^2)\right]^2}$$

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem:

$$\angle T(j\omega) = -\tan^{-1}\left(\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right)$$

$$\frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega} = \frac{F_1(\omega^2)\left[\frac{d\left(\omega F_2(\omega^2)\right)}{d\omega}\right] - \left(\omega F_2(\omega^2)\right)\frac{d\left(F_1(\omega^2)\right)}{d\omega}}{\left[F_1(\omega^2)\right]^2}$$

Odd

Even

Even

Thus this term is an even rational fraction in ω

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem:

$$\tau_G = -\frac{d\angle T(j\omega)}{d\omega} = -\frac{1}{1+\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]^2} \bullet \frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega}$$

Even

It follows that τ_G is the product of rational fractions in ω^2 so it is also a rational fraction in ω^2

Although tedious, the results can be extended when there are zeros present in T(s) as well

Thomson and Bessel Approximations

- All-pole filters
- Maximally linear phase at ω=0
- $$\left.\frac{d\angle T(j\omega)}{d\omega}\right|_{\omega=0} = -1$$ (this is normalized to phase derivative = -1 at ω=0 and is the counterpart to normalizing a band edge in a lowpass filter to ω=1)

since $$\tau_G = -\frac{d\angle T(j\omega)}{d\omega}$$

These criteria can be equivalently expressed as

- All-pole filters
- Maximally constant group delay at ω=0
- $\tau_G = 1$ at ω=0 (this is normalized to τ_G =1 at ω=0 and is the counterpart to normalizing a band edge in a lowpass filter to ω=1)

Thomson and Bessel Approximations

$$T_A(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

Must find the coefficients a_0, a_1,… a_n to satisfy the maximal constant group delay constraints

$$T(j\omega) = \frac{1}{\left(1 - a_2\omega^2 + a_4\omega^4 + \ldots\right) + j\omega\left(a_1 - a_3\omega^2 + a_5\omega^4 + \ldots\right)}$$

Theorem: If $T(j\omega) = \frac{1}{x + jy}$ then τ_G is given by the expression

$$\tau_G = \frac{x\frac{dy}{d\omega} - y\frac{dx}{d\omega}}{x^2 + y^2}$$

This theorem is easy to prove using the identity given above, proof will not be given here

Stay Safe and Stay Healthy !

End of Lecture 11